Analysis of Shear Stress on Flow around Sudden Accelerated Plate (Stokes' First Problem)

Andi Firdaus Sudarma

Department of Mechanical Engineering, Universitas Mercu Buana, Meruya Selatan, Kembangan, Jakarta Barat 11680, Indonesia

Corresponding Author Email: andi.firdaus@mercubuana.ac.id

ARTICLE HISTORY

ABSTRACT

Stokes' First Problem, often referred to as the "sudden accelerated plate," was studied using similarity method to obtain velocity and shear stress profile by analyzing the flow of an infinite body of fluid near a wall that experiences sudden motion. The flow is assumed to be Newtonian, viscous, and incompressible, while at initial condition the velocity considered as zero and the condition of the flow were at rest. The obtained results are then numerically solved employing Simpson's approximation. Furthermore, this study explores variations in velocity and shear stress at the wall across different time intervals. The study of the velocity profile within this scenario demonstrates its consistency with the non-slip condition and the specified boundary conditions. Specifically, for t > 0, the velocity of the flow at the surface (y = 0) aligns with the plate's speed, while at y = ∞, the velocity decreases to zero, mirroring the initial condition. The findings reveal that at the moment the plate initiates its motion (t = 0), the shear stress reaches its maximum value. As time progresses, the shear stress at the wall gradually decreases.

1. INTRODUCTION

Stokes' First Problem, often referred to as the "sudden accelerated plate," stands as a classic scenario in fluid dynamics, holding profound significance in the study of viscous flows. Notably, the application of the separation variable method to solve this problem can lead to physically incorrect solutions. In contrast, the similarity formulation, initially addressed by G. Stokes, is widely recognized as a more suitable approach for tackling this specific problem (Stokes, 1851).

In this context, the problem involves the abrupt motion of a semi-infinite flat plate within a viscous fluid. This motion commences from a state of rest and accelerates rapidly to achieve a constant velocity. Such a phenomenon mirrors the behavior observed on the surface of aircraft wings during takeoff. The plate's motion generates a flow field imbued with intriguing characteristics, which serves to illuminate the fundamental behavior of viscous fluids subjected to sudden changes in boundary conditions (Bird et al., 1961).

Stokes' First Problem serves as a cornerstone in understanding the dynamics of fluid motion, offering valuable insights into the intricate interplay between viscosity, inertia, and boundary conditions. It significantly contributes to our understanding of boundary layer development, the establishment of velocity profiles, and the emergence of flow instabilities. Through meticulous analysis, experimental, and numerical simulations, researchers have gained a deeper appreciation of the complexities inherent in fluid flows and the pivotal role played by fundamental principles in elucidating these phenomena (Grift et al., 2019; Joshi & Bhattacharya, 2022).

While the classic Stokes' problem remains of enduring interest to researchers, various modifications have been explored (Liu, 2008a, 2008b; Welker, 2019). One such adaptation involves the study of magneto-hydrodynamics flow over a suddenly accelerated flat

plate. Researchers like Haque and Zaman have approached this problem, utilizing both analytical and numerical methods, to gain a deeper understanding (Haque & others, 2018; Zaman et al., 2014). Recent studies, researchers have ventured into investigating the behavior of flat plates that suddenly move through a yield-stress material and also a viscous incompressible fluid (Hinton et al., 2022; Kharchandy, 2018). However, the shear stress behavior on this problem was not much discussed in previous research. In this context, the research into shear stress in the flow remains compelling, especially when the plate begins its motion.

This research, specifically, delves into the behavior of velocity in the vicinity of a suddenly accelerated flat plate. The velocity profile is obtained using the similarity method, as outlined in Stokes' first problem, resulting in a single ordinary differential equation (Schlichting, 1979). The obtained results are then numerically solved employing Simpson's approximation. Furthermore, this study explores variations in velocity and shear stress at the wall across different time intervals, contributing to the evolving understanding of this dynamic scenario.
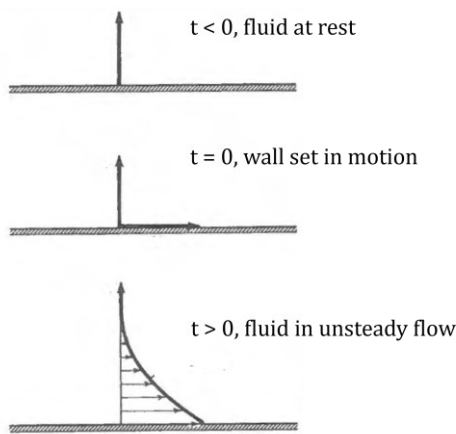

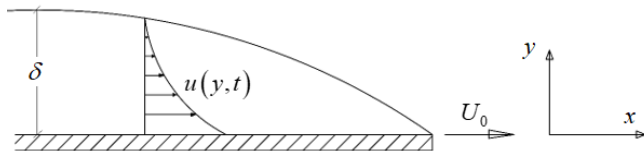Fig. 1. The viscous flow of fluid near a wall suddenly set in motion.

Fig. 2. Stokes' first problem; a stationary plate suddenly moved along x axis.

2. METHODS

2.1 Similarity Solution Method

The unsteady viscous flow problem of a suddenly accelerated plate is the focus of this study. To elucidate, we consider a scenario in which the plate is initially stationary at $t = 0$, and subsequently, at $t > 0$, it commences motion within its own plane, along the x-direction, attaining a constant velocity denoted as U_o (as shown in Fig. 1 and Fig. 2).

Obtaining the solution begins by analyzing the flow of an infinite body of fluid near a wall that experiences sudden motion. This problem provides a compelling opportunity to showcase the utility of the similarity method, a mathematical approach that allows us to transform a complex partial differential equation into a more manageable single ordinary differential equation. This transformation simplifies the mathematical treatment of the problem and is an essential step in gaining a deeper understanding of the dynamics involved.

Continuity equation

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0 \tag{1}$$

Momentum equation in x-direction

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial P}{\partial x} + v\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \tag{2}$$

Momentum equation in y-direction

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{\partial P}{\partial y} + v\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \tag{3}$$

In the context of sudden accelerated flat plate flow, it is reasonable to assume that the velocity in the y-direction is negligible, especially when the flow is fully developed, and the pressure remains constant. Under these conditions, the governing equations can be significantly simplified, resulting in the following equation:

$$\frac{\partial u}{\partial t} = v\frac{\partial^2 u}{\partial y^2} \tag{4}$$

Since the plate is considered infinite, and the fluid domain is semi-infinite, there is no inherent geometric length scale that affects the solution. Consequently, employing dimensional analysis yields the following result:

$$\frac{u}{U_o} = f(v, y, t) \tag{5}$$

The initial boundary conditions and assumptions for this study are as follows:
1. The flow is assumed to be Newtonian, viscous, and incompressible (Ismoen et al., 2015).
2. Initial Condition: At t < 0, the velocity component u is set to zero for all values of y.
3. Boundary Condition: at y=0, u=Uo for all t>0 and at y=0 for all t>0.

Assume that the solution in form of

$$u(y, t) = \phi(t)f(\eta) \tag{6}$$

Where $\eta = \frac{y}{g(t)}$
and;
$\varphi(t)$ = function of t (time) only
$g(t)$ = scaling function of t

$f(\eta)$ = non dimensional velocity in form of function (non-dimensional coordinate)

To solve Equation 6 effectively, it is necessary to define each term within it. Subsequently, the equation can be substituted to derive the similarity solution, facilitating a deeper understanding of the problem at hand. More detailed derivation of the equation can be found in the (Sudarma, 2012).

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial t}(\varphi(t)f(\eta)) \tag{7}$$

Through the application of the chain rule, the equation above can be expressed as follows:

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial t}(\varphi(t)f(\eta)) = \phi'(t)f(\eta) + \phi(t)\frac{\partial f'(\eta)}{\partial \eta}\frac{\partial \eta}{\partial t} \tag{8}$$

where,

$$\frac{\partial \eta}{\partial t} = \frac{\partial}{\partial t}\left(\frac{y}{g(t)}\right) = -\eta\frac{g'(t)}{g(t)} \tag{9}$$

By applying boundary condition, which states that $\phi(t) = U_0$ and $\phi'(t) = 0$, to equation 4, the equation undergoes simplification, resulting in:

$$\frac{\partial u}{\partial t} = -U_0 f'(\eta)\eta\frac{g'(t)}{g(t)} \tag{10}$$

When differentiating the above equation with respect to time (t), we obtain:

$$\frac{\partial^2 u}{\partial t^2} = \frac{U_0}{g^2(t)}f''(\eta) \tag{11}$$

Integrating equation 10 with respect to time (t) will yield:

$$u = U_0 f(\eta) \tag{12}$$

By substituting equations 11, 12, and 13 into equation 4, the solution can be written as follows.

$$f''(\eta) + \frac{g'(t)g(t)}{v}\eta f'(\eta) = 0 \tag{13}$$

By defining a as $\frac{g'(t)g(t)}{v}$ and applying integration with respect to t obtaining the following.

$$\frac{g^2(t)}{2v} + C = a.t \tag{14}$$

$$g(t) = \sqrt{2avt} \quad \text{where C = 0}$$

Assuming a value of a = 2 and substituting it into equation 14, the derivation of the following ordinary differential equation for η:

$$f''(\eta) + 2\eta f'(\eta) = 0 \tag{15}$$

To solve the equation above, let's introduce f'(η)=S, and f''(η)=S'. This transformation allows the solution to be expressed as the following expression:

$$S' + 2\eta S = 0 \tag{16}$$

Upon integrating both sides with respect to η, it will yield:

$$\frac{S}{C} = e^{-\eta^2} \tag{17}$$

$$S = f'(\eta) = Ce^{-\eta^2} \tag{18}$$

And integrating it again with respect to η

$$f(\eta) = C_1 \int_0^\eta e^{-\eta^2}\, d\eta + C_2 \tag{19}$$

Where initially η = 0 has been chosen as the lower limit of the indefinite integral, it acknowledged that this integral cannot be evaluated in closed form. Changing the lower limit from η = 0 to another value would merely alter the constant C_2, which remains undetermined. However, by applying the specified boundary conditions and subsequently evaluating equation 19, the values of C_1 and C_2 can be determined.

This equation is to be solved under the following initial and boundary conditions relevant to the problem: f(0) = 1 and f(∞) = 0.

$$C_2 = 1 \quad \text{and} \quad C_1 = -\frac{1}{\int_0^\infty e^{-\eta^2}\, d\eta} \tag{20}$$

By substituting the constants, the equation 19 can be expressed as follows:

$$f(\eta) = 1 - \frac{2}{\sqrt{\pi}}\int_0^\eta e^{-\eta^2}\, d\eta \tag{21}$$

Or it can be expressed as error function (Cody, 1993)

$$\frac{u}{U_o} = 1 - erf\left(\frac{y}{2\sqrt{vt}}\right) \tag{22}$$

2.2 Simpson's Rule Integration Method

To determine the values of velocity, f(η), we perform numerical integration of equation 21 using Simpson's approximation. The integration, represented in the form of $\int_a^b f(x)dx$, can be solved using the following formula (Cartwright, 2017)

$$S_n = \frac{\Delta x}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \ldots + 2y_{n-2} + 4y_{n-1} + y_n) \tag{23}$$

Where $\Delta x = (b-a)/n$, and n is an even arbitrary number.

3. RESULTS AND DICUSSION

In the analysis of the sudden accelerated plate case, current research employed the similarity method to formulate the problem and then numerically integrated it using Simpson's rule. To solve Equation 22 with Simpson's rule, as outlined in Equation 24, the calculation

set n to 10, resulting in $\Delta x = \eta/10$. The assumption is that the value of η, representing the maximum limit of the integral in Equation 22, ranges from 0 to 2, leading to Δx being 0.2.
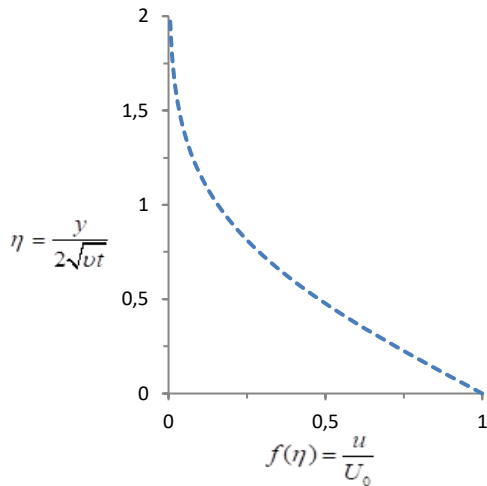

Fig. 3. Velocity distribution above a suddenly accelerated plate.

Figure 3 presents the velocity profile, indicating that at the plate surface ($\eta = 0$), both the flow and the plate share the same velocity, while for the flow far away from the surface ($\eta = \infty$), the velocity reaches zero. This observation aligns with the non-slip condition and the specified boundary conditions. Specifically, for t > 0, the velocity of the flow at the surface (y = 0) equals the plate's speed, and at y = ∞, the velocity reduces to zero. Therefore, it is evident that Figure 3 and the boundary conditions yield consistent velocity profiles.

Fig. 3 reveals that as the plate initiates its motion, a boundary layer forms near its surface due to the no-slip condition. This boundary layer is characterized by a gradient of fluid velocity, transitioning from the stationary surface of the plate to the free stream velocity of the fluid (as seen in Fig. 2). Over time, the thickness of the boundary layer increases as it adapts to the plate's acceleration. The evolution of the flow field is influenced by the diffusion of momentum within the fluid, gradually establishing a parabolic velocity profile across the boundary layer.

Figure 4 provides a comparison of the velocity profiles at different times, with y representing the boundary layer thickness (as labeled δ in Figure 2). As time progresses, the boundary layer thickness increases. Notably, the velocity profiles for different times exhibit a 'similar' trait. In other words, they can be scaled to a common curve along the ordinate axis.

Figure 4 illustrates the velocity distribution, and it's noteworthy that the velocity profiles at different times exhibit a 'similar' characteristic. In essence, these profiles can be normalized to a common curve by adjusting the scale along the ordinate axis.

By substituting equation 9 into equation 15, the boundary layer thickness is expressed as follow,

$$y = 2\eta\sqrt{vt} \tag{24}$$

Assume that the surrounding fluid is air at temperature 20°C with atmospheric pressure. The properties of the corresponding fluid are available in the reference (Fox et al., 2011).

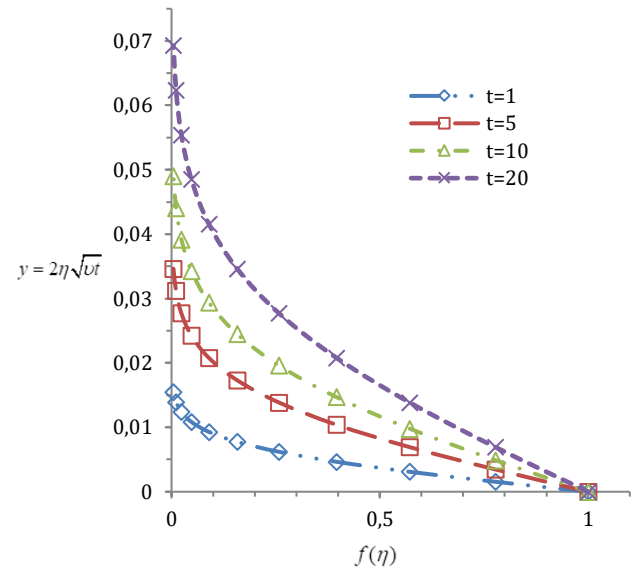For time are varied, where t = 1, 5, 10 and 20, the velocity distribution is represented in Figure 4.

Fig. 4. Velocity Profile for Varies Time.

To obtain the shear stress at the wall, τ_w, the general form was expressed by the following,

$$\tau_w = \mu \left.\frac{\partial u}{\partial y}\right|_{y=0} \tag{25}$$

We begin with equation 19 and then substitute with equation 13.

$$u(y,t) = U_0\left(1 - \frac{2}{\sqrt{\pi}}\int_0^\eta e^{-\eta^2}\, d\eta\right)$$
$$= U_0 - \frac{2U_0}{\sqrt{\pi}}\int_0^\eta e^{-\eta^2}\, d\eta \tag{26}$$

The derivative of equation above with respect to *y* is given below,

$$\frac{\partial u}{\partial y} = \frac{2U_0}{\sqrt{\pi}}\frac{e^{-\eta^2}}{g(t)} \tag{27}$$

Substituting equation above into equation 25 by including $g(t) = 2\sqrt{vt}$, gives

$$\tau_w = \mu \left.\frac{2U_0}{\sqrt{\pi}}\frac{e^{-\eta^2}}{2\sqrt{vt}}\right|_{y=0} \tag{28}$$

Where $v = \mu/\rho$.
By simplifying the equation, it can be expressed as

$$\tau_w = U_0\frac{\mu}{\sqrt{\pi v t}} \quad (N/m^2) \tag{29}$$

Solving the first Stokes' problem using Laplace transform has been done previously in other research where the shear stress influenced by several parameters including fluid density (Kharchandy, 2018). In comparison, shear stress result in equation 29 where the effect of viscosity, μ, and density, ρ.

Examining the shear stress at the wall, as demonstrated in Figure 5 for various time instances, reveals that at this moment the plate initiates its motion (t = 0), the shear stress reaches its maximum value. As time advances, the shear stress at the wall gradually decreases.

For time are varied, where t = 1, 5, 10 and 20 and   is constant, the shear stress at wall is represented in Figure 5.
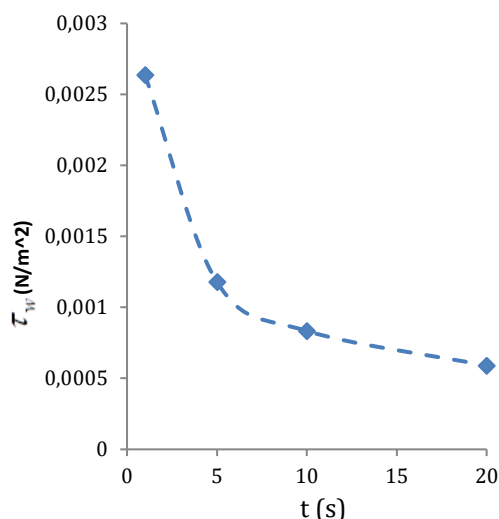

Fig. 5. Shear stress at the plate wall for varied time.

4. CONCLUSION

The current research involves an analysis of the sudden accelerated plate case, primarily employing the similarity method. The study of the velocity profile within this scenario demonstrates its consistency with the non-slip condition and the specified boundary conditions. Specifically, for t > 0, the velocity of the flow at the surface (y = 0) aligns with the plate's speed, while at y = ∞, the velocity decreases to zero, mirroring the initial condition. The findings reveal that at the moment the plate initiates its motion (t = 0), the shear stress reaches its maximum value. As time progresses, the shear stress at the wall gradually decreases.

Modification of this problem is challenging and attracts many researchers' interests recently. Applying other powerful mathematical methods may contribute to simplifying the solution.

5. ACKNOWLEDGMENTS

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